ON STRONGLY EXPOSED POINTS AND FRECHET DIFFERENTIABILITY†

BY

J. COLLIER AND M. EDELSTEIN

ABSTRACT

Using properties of convex functionals, it is shown that closed and bounded convex sets in a class of Banach spaces which includes separable conjugate spaces are the closed convex hulls of their strongly exposed points.

Bessaga and Pełczyński [4] proved that each closed, bounded, convex subset of a separable, conjugate Banach space is the closed, convex hull of its extreme points. Asplund [2] has observed that this is also true for a larger class of Banach spaces, for the conjugate of a strong differentiability space (a term first introduced by Asplund in [1]). This follows from his result [1] that each weak*-compact convex subset of the conjugate of a strong differentiability space is the weak*-closed, convex hull of its weak*-strongly exposed points. The results of the present paper imply a corresponding statement for closed and bounded convex sets.

THEOREM 1. *Suppose X is a Banach space which is the conjugate of a strong differentiability space; e.g., a separable conjugate space. Then each closed, bounded, convex subset of X is the closed, convex hull of its strongly exposed points.*

A point p in a subset C of a Banach space X is called strongly exposed if there is an $f \in X^*, f \neq 0$, such that $f(p) = \sup f[C]$ and for any sequence $\{x_n\} \subseteq C$, $x_n \to p$ in norm whenever $f(x_n) \to f(p)$. A strongly exposed point of C is necessarily an exposed point of C; that is, $f(p) > f(q)$ for all $q \in C \setminus \{p\}$. Troyanski [5] has shown that a weakly compact, convex subset of any Banach space is the closed, convex hull of its strongly exposed points.

† This research was supported by the National Research Council of Canada, Grant A-3999.
Received March 22, 1973

A convex function f on a Banach space X is continuous on the interior of $\{x \in X \mid f(x) < +\infty\}$. A strong differentiability space (SDS) is a Banach space for which each convex function continuous on the space is Fréchet differentiable on a dense G_δ subset. We call a conjugate Banach space a weak*-SDS if each continuous, convex function which is also weak*-lower semicontinuous is Fréchet differentiable on a dense G_δ subset. Each of the following spaces has been shown to be an SDS: A Banach space whose conjugate is separable [1]; $c_0(\Gamma)$, where Γ is an arbitrary index set [2]; a reflexive Banach space [5].

We recall the basic relationship between strongly exposed points and Fréchet differentiability. The complete details may be found in [3]. If f is a convex function on a Banach space X, then its conjugate on X^* is the (weak*-lower semicontinuous) convex function defined by $f^*(y) = \sup\{y(x) - f(x) \mid x \in X\}$ for $y \in X^*$. We say that f is norm rotund at $a \in X$ relative to $b \in X^*$ if $f^*(b) = b(a) - f(a)$ and for any $\{x_n\} \subseteq X$, $x_n \to a$ in norm whenever $b(x_n) - f(x_n) \to b(a) - f(a)$. If $f(x) = 0$ for each x in some subset C of X and $f(x) = +\infty$ for $x \notin C$, then f is called the indicator function on C and the point $a \in C$ is strongly exposed by $b \in X^*$ if and only if f is norm rotund at a relative to b. If f and f^* are convex functions (not identically $+\infty$) on X and X^*, respectively, which are conjugate to each other, then f is norm rotund at a relative to b if and only if f^* is Fréchet differentiable at b relative to a (that is, with Fréchet gradient a) [3]. In each of the preceding statements X and X^* may be interchanged.

PROPOSITION 1. *Let X be a Banach space. If X is an* SDS, *then X^{**} is a weak*-SDS.*

PROOF. Suppose f^* is a continuous, convex function on X^{**} which is also weak*-lower semicontinuous. Let T be an arbitrary isomorphism of X^* onto itself. Let f be the weakly lower semicontinuous convex function on X^* which is conjugate to f^*. Let g be the continuous convex function on X which is conjugate to fT^{-1}. Let g^* be the weak*-lower semicontinuous convex function on X^* which is conjugate to g. Then f and f^*, g and g^*, fT^{-1} and f^*T^* are pairs of mutually conjugate convex functions, $g = f^*T^*/X$ where we consider $X \subseteq X^{**}$, and g^* is the weak*-lower, semicontinuous, convex hull of fT^{-1}.

If g^* is norm rotund at $b \in X^*$ relative to $a \in X$, then from [3, proof of Prop. 2], fT^{-1} is also norm rotund at b relative to a. Moreover, it follows easily from the definition of norm rotundity that fT^{-1} is norm rotund at b relative to a if and only if f is norm rotund at $T^{-1}b$ relative to T^*a, where we consider $X \subseteq X^{**}$.

Suppose g is Fréchet differentiable at some $a \in X$ relative to $b \in X^*$. Then g^* is norm rotund at b relative to a. Hence fT^{-1} is also norm rotund at b relative to a, which implies that f is norm rotund at $T^{-1}b$ relative to T^*a. Thus f^* is Fréchet differentiable at T^*a relative to $T^{-1}b$. Since X is an SDS, g is Fréchet differentiable on a dense subset of X; whence f^* is Fréchet differentiable on a dense subset of $T^*[X]$. However, T was chosen as an arbitrary isomorphism of X^* onto itself and it may be easily shown that for any point $p \in X^{**}$, there is some isomorphism T for which $p \in T^*[X]$. Thus the set D of points at which f^* is Fréchet differentiable is dense in X^{**}. It follows from [1, Lem. 6] that D is in fact a dense G_δ subset of X^{**}. ∎

PROPOSITION 2. *Let X be the conjugate of an* SDS *and suppose C is a bounded, norm-closed subset. Then the set of all functionals in X^* which strongly expose some point of C is a dense G_δ.*

PROOF. Let $\bar{C}$ be the closed, convex hull of C, let f be the indicator function of $\bar{C}$, and let $f^*(y) = \sup\{y(x) \mid x \in \bar{C}\}$, $y \in X^*$, be the support function of $\bar{C}$. Then f and f^* are convex functions which are conjugate to each other. Since f^* is a weak*-lower semicontinuous convex function continuous on X^*, the set D of points at which f^* is Fréchet-differentiable is a dense G_δ subset of X^*. Moreover, each Fréchet gradient of f^* (by definition in X^{**}) actually belongs to X [3]. Thus each $y \in D$ is a functional which strongly exposes some point in $\bar{C}$. If $y \in D$ strongly exposes $x \in \bar{C}$, then there is a sequence $\{x_n\} \subseteq C$ such that $y(x_n) \to y(x)$ and hence $x_n \to x$ in norm. Since C is closed, $x \in C$. It follows that each $y \in D$ strongly exposes some point of C.

Theorem 1 follows immediately from Proposition 2.

REFERENCES

1. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. **121** (1968), 31–47.

2. E. Asplund, *Boundedly Krein-compact Banach spaces*, Proceedings of the Functional Analysis Week, Aarhus, 1969, pp. 1–4.

3. E. Asplund, and R. T. Rockafellar, *Gradients of convex functions*, Trans. Amer. Math. Soc. **139** (1969), 344–467.

4. C. Bessaga and A. Pełczyński, *On extreme points in separable conjugate spaces*, Israel J. Math. **4** (1966), 252–264.

5. S. L. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.

DEPARTMENT OF MATHEMATICS
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA, CANADA